Math Methods for Economics

Summer 2019 Midterm Exam

Answer all seven questions on blank letter-size papers. Show **ALL** work, including intermediate steps. Failure to show intermediate steps will result in zero credit. Please write your name on your exam.

1. Find the first derivative of the following function.

$$f(x) = \frac{e^{2x^2-4x+3} \cdot ln(2x^2-4x+3)}{x}$$

Solution:

Apply the chain rule and product rule to find the derivative of the numerator.

$$(4x-4)e^{2x^2-4x+3}\ln(2x^2-4x+3)+e^{2x^2-4x+3}\frac{4x-4}{2x^2-4x+3}$$

The derivative of the denominator is = 1.

Given the above, we can find the derivative of f(x) using the quotient rule:

$$f'(x) = \frac{(4x-4)e^{2x^2-4x+3}\ln\left(2x^2-4x+3\right)+e^{2x^2-4x+3}\frac{4x-4}{2x^2-4x+3}-\left(e^{2x^2-4x+3}\cdot\ln(2x^2-4x+3)\right)}{x^2}$$

2. Using the following function, please
   (a) Find the first derivative.
   (b) Determine the location of stationary points.
   (c) Determine whether each point is a local minimum, maximum or an inflection point using the second derivative. Please be sure to say how you have determined it is a local maximum, minimum or inflection point.

$$f(x) = \frac{e^{4x^3}}{e^{-2x^2}}e^{-x^3}$$

Solution:

(a) First, we can simplify the expression of the function by using the power law.

$$f(x) = e^{4x^3-x^3+2x^2} = e^{3x^3+2x^2}$$

Then we can calculate out the first derivative

$$f'(x) = (9x^2+4x)\times e^{3x^3+2x^2}$$

(b) First Order Condition $f'(x) = 0$

Since exponential function is always > 0, the FOC holds if and only if $9x^2+4x = 0$.

Thus, $x = 0$ and $-4/9$ are the location of stationary points.

(c) Second order Sufficient Condition Test

$$f''(x) = (18x+4)e^{3x^3+2x^2} + (9x^2+4x)e^{3x^3+2x^2}$$

$$f''(0) = 4 > 0$$

When x = 0, second order derivative is positive, it is a minimum.

When $x = -4/9$, $f''(-4/9) < 0$. Because the second derivative is negative we know that it is a maximum.

3. Solve the following system of equations using Cramer's Rule.

$$4x + 3y - 2z = 7$$

$$x + y = 5$$

$$3x + z = 4$$

Solution:

Convert the system into a matrix form will be a lot easier.

$$\begin{bmatrix} 4 & 3 & -2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 4 \end{bmatrix}$$

Apply Cramer's Rule

$$x = \frac{\begin{vmatrix} 7 & 3 & -2 \\ 5 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 & -2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{7+8-15}{4+6-3} = \frac{0}{7} = 0$$

$$y = \frac{\begin{vmatrix} 4 & 7 & -2 \\ 1 & 5 & 0 \\ 3 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 & -2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{20-8+30-7}{4+6-3} = \frac{35}{7} = 5$$

$$z = \frac{\begin{vmatrix} 4 & 3 & 7 \\ 1 & 1 & 5 \\ 3 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 4 & 3 & -2 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}} = \frac{16+45-21-12}{4+6-3} = \frac{28}{7} = 4$$

4. Evaluate the following definite integral

$$\int_0^1 \frac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}} dx$$

Solution:
Apply the rules of change of variables. Let $u = e^{2x} + e^{-2x}$,

Then

$$\frac{du}{dx} = 2(e^{2x} - e^{-2x})$$

Solving for dx gives

$$dx = \frac{du}{2(e^{2x} - e^{-2x})}$$

Inserting the above and simplifying the original integral becomes

$$\int \frac{1}{u} du = \ln u = \ln(e^{2x} + e^{-2x})$$

Evaluate from $x = 0$ to $x = 1$,

$$\ln\left(e^{2x} + e^{-2x}\right)\Big|_{x=0}^{1} = \ln\left(e^{2} + e^{-2}\right) - \ln 2$$

$$= \ln\left(\frac{e^{2} + e^{-2}}{2}\right)$$

5. Solve the three unknown variables x, y, z using the matrix inversion technique discussed in this course.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 9 \end{bmatrix}$$

Solution:
Let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{bmatrix}$$

Then the original system becomes

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 9 \end{bmatrix}$$

Find the minors and cofactors.

$$\text{Principal Minor } A = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

Co-factor A

$$\text{Cofactor } A = \begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -5 & 4 & -1 \\ 3 & -3 & 0 \\ 4 & -5 & 2 \end{bmatrix}$$

Adjoint A (transpose of Co-factor)

$$\begin{bmatrix} -5 & 3 & 4 \\ 4 & -3 & -5 \\ -1 & 0 & 2 \end{bmatrix}$$

Now we need to find the value of the Determinant.

$$D = 2(-5) + 2(4) + 1(-1) = -3$$

To find the inverse we multiple the adjoint by $\frac{1}{\text{detA}}$.

$$A^{-1} = -\frac{1}{3}\begin{bmatrix} -5 & 3 & 4 \\ 4 & -3 & -5 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -5 & 3 & 4 \\ 4 & -3 & -5 \\ -1 & 0 & 2 \end{bmatrix} \cdot 3 \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$$

6. Consider a function $y = f(x) = ax^3 + bx^2 + cx$. At the point (1,5), the values of the first and second derivative are -1 and 0 respectively. Use this information to find the values of coefficient a, b, c.

Solution:

The question mentions three conditions.

The point (1,5) is on the curve. So

$$f(1) = 5$$

Thus we have

$$a + b + c = 5$$

The slope of tangent line at (1,5) is equal to -1. Thus,

$$f'(1) = -1$$

$$3a + 2b + c = -1$$

Lastly, the the second derivative value is equal to zero at (1,5). Thus,

$$f''(1) = 0$$

$$6a + 2b = 0$$

We know have three equations and three unknowns.

$$a + b + c = 5$$

$$3a + 2b + c = -1$$

$$6a + 2b = 0$$

Solve out the system of equations, we find

$$a = 6, b = -18, c = 17$$

That is,

$$f(x) = 6x^2 - 18x + 17$$

7. Solve for following indefinite integral.

$$\int \frac{\sqrt[3]{\ln x}}{x} dx$$

Solution:

Let u = lnx

$$\frac{du}{dx} = \frac{1}{x}$$

Solving for dx gives:

$$dx = x * du$$

Inserting the defintion of u and dx gives

$$\int \frac{\sqrt[3]{u}}{x} x * du$$

Cancelling terms gives

$$\int \sqrt[3]{u} du = \frac{3}{4} u^{\frac{4}{3}} + C$$

Where u = lnx, so the answer is

$$\int \frac{\sqrt[3]{\ln x}}{x} dx = \frac{3}{4} (\ln x)^{\frac{4}{3}} + C$$